WEAK $\mathcal{Z}$-STRUCTURES FOR SOME CLASSES OF GROUPS

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Abstract. Motivated by the usefulness of boundaries in the study of δ-hyperbolic and CAT(0) groups, Bestvina introduced a general approach to group boundaries via the notion of a $\mathcal{Z}$-structure on a group G. Several variations on $\mathcal{Z}$-structures have been studied and existence results have been obtained for some very specific classes of groups. However, little is known about the general question of which groups admit any of the various $\mathcal{Z}$-structures, aside from the (easy) fact that any such G must have type F, i.e., G must admit a finite $\mathrm{K}(G,1)$. In fact, Bestvina has asked whether *every* type F group admits a $\mathcal{Z}$-structure or at least a "weak" $\mathcal{Z}$-structure.

In this paper we prove some rather general existence theorems for weak $\mathcal{Z}$-structures. Among our results are the following:

Theorem A. *If nontrivial groups H_1 and H_2 have type F, then $H_1 \times H_2$ admits a weak $\mathcal{Z}$-structure.*

Theorem B. *If G admits a finite $K(G,1)$ complex K such that the corresponding G-action on $\widetilde{K}$ contains $1 \neq j \in G$ properly homotopic to $\mathrm{id}_{\widetilde{K}}$, then G admits a weak $\mathcal{Z}$-structure.*

Theorem C. *If H has type F, then every semidirect product $H \rtimes_{\phi} \mathbb{Z}$ admits a weak $\mathcal{Z}$-structure.*

Theorem D. *If G has type F and is simply connected at infinity, then G admits a weak $\mathcal{Z}$-structure.*

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1. Introduction

Several lines of investigation in geometric topology and geometric group theory seek 'nice' compactifications of contractible manifolds or complexes (or ERs/ARs) on which a given group G acts cocompactly as covering transformations. Bestvina [Be] has defined a $\mathcal{Z}$-structure and a weak $\mathcal{Z}$-structure on a group G as follows:

A *$\mathcal{Z}$-structure* on a group G is a pair $\left(\overline{X}, Z\right)$ of spaces satisfying:

(1) $\overline{X}$ is a compact ER,
(2) Z is a $\mathcal{Z}$-set[1] in $\overline{X}$,
(3) $X = \overline{X} - Z$ admits a proper, free, cocompact action by G, and
(4) (nullity condition) For any open cover $\mathcal{U}$ of $\overline{X}$, and any compactum $K \subseteq X$, all but finitely many G-translates of K lie in some element U of $\mathcal{U}$.

If only conditions 1)-3) are satisfied, $\left(\overline{X}, Z\right)$ is called a *weak $\mathcal{Z}$-structure* on G.

An additional condition that can be added to conditions 1)-3), with or without condition 4), is:

(5) The action of G on X extends to an action of G on $\overline{X}$.

Farrell-Lafont refer to a pair $\left(\overline{X}, Z\right)$ satisfying 1)-5) as an *$\mathcal{EZ}$-structure.* Others have considered pairs that satisfy 1)-3) and 5); we call those *weak $\mathcal{EZ}$-structures.* Depending on the set of conditions satisfied, Z is referred to generically as a *boundary* for G; or more specifically as a *$\mathcal{Z}$-boundary*, a *weak $\mathcal{Z}$-boundary,* an *$\mathcal{EZ}$-boundary*, or a *weak $\mathcal{EZ}$-boundary.*

Example 1. *Torsion-free CAT(0) groups admit $\mathcal{EZ}$-structures—one compactifies a corresponding CAT(0) space by adding the visual boundary. Bestvina-Mess* [BM] *have shown that each torsion-free word hyperbolic group G admits an $\mathcal{EZ}$-structure $\left(\overline{X}, \partial G\right)$, where ∂G is the Gromov boundary and X is a Rips complex for G. Osajda and Przytycki have shown that systolic groups admit $\mathcal{EZ}$-structures.* [Be] *contains a discussion of $\mathcal{Z}$-structures and weak $\mathcal{Z}$-structures on a variety of other groups, not all of which satisfy condition 5).*

Remark 1. Some authors (see [Dr]) have extended the above definitions by allowing non-free G-actions (thus allowing for groups with torsion) and by loosening the ER requirement on $\overline{X}$ to that of AR, i.e., allowing $\overline{X}$ to be infinite-dimensional. Here we stay with the original definitions, but note that some analogous results are possible in the more general settings.

A group G has *type F* if it admits a finite $\mathrm{K}(G, 1)$ complex. The following proposition narrows the field of candidates for admitting any sort of $\mathcal{Z}$-structure to those groups of type F.

Proposition 1.1. *If there exists a proper, free, cocompact G-action on an AR Y, then G has type F.*

[1] The definition of $\mathcal{Z}$-set, along with definitions of numeous other terms used in this introduction, can be found in §2.

Proof. The quotient $q : Y \to G\backslash Y$ is a covering projection, so $G\backslash Y$ is aspherical and locally homeomorphic to Y. By the latter, $G\backslash Y$ is a compact ANR, and thus (by Theorem 2.1) homotopy equivalent to a finite complex. Any such complex is a $\mathrm{K}(G,1)$. □

In [Be], Bestvina asked the following pair of questions:

Bestvina's Question. *Does every type F group admit a $\mathcal{Z}$-structure?*

Weak Bestvina Question. *Does every type F group admit a weak $\mathcal{Z}$-structure?*

The Weak Bestvina Question was also posed by Geoghegan in [Ge2, p.425]. Farrell and Lafont [FL] have asked whether every type F group admits an $\mathcal{EZ}$-structure, and the question of which groups admit weak $\mathcal{EZ}$-structures appear in both [BM] and [Ge2]. Although interesting special cases abound, a general solution to any of these questions seems out of reach at this time.

As one would expect, the more conditions a $\mathcal{Z}$-structure or its corresponding boundary satisfies, the greater the potential applications. For example, Bestvina has shown that the topological dimension of a $\mathcal{Z}$-boundary is an invariant of the group—it is one less than the cohomological dimension of G; this is not true for weak $\mathcal{Z}$-boundaries. But a weak $\mathcal{Z}$-boundary carries significant information about G. For example, the Čech cohomology of a weak $\mathcal{Z}$-boundary reveals the group cohomology of G with $\mathbb{Z}G$-coefficients, and the pro-homotopy groups of a weak $\mathcal{Z}$-boundary are directly related to the corresponding end invariants (such as simple connectivity at infinity) of G. A weak $\mathcal{Z}$-boundary, when it exists, is well-defined up to shape and can provide a first step toward obtaining a stronger variety of $\mathcal{Z}$-structure on G. $\mathcal{EZ}$- and weak $\mathcal{EZ}$-boundaries, when they exist, carry the potential for studying G by analyzing its action on the compactum Z. More about these topics can be found in [Ge1], [Ge2], [GM1], [Be], [FL] and [Gu3].

In this paper we prove the existence of weak $\mathcal{Z}$-structures for a variety of groups. A notable special case provides a "stabilized solution" to the Weak Bestvina Question. It asserts that, if H has type F, then $H \times \mathbb{Z}$ admits a weak $\mathcal{Z}$-structure. That result is an easy consequence of either of the following more general theorems, to be proven here.

Theorem 1.2. *If H_1 and H_2 are nontrivial type F groups, then $H_1 \times H_2$ admits a weak $\mathcal{Z}$-structure. More generally, if a type F group G is virtually a product of two nontrivial type F groups, then G admits a weak $\mathcal{Z}$-structure.*

Theorem 1.3. *Suppose G admits a finite $K(G,1)$ complex K, and the corresponding G-action on the universal cover $\widetilde{K}$ contains a $1 \neq j \in G$ that is properly homotopic to $\mathrm{id}_{\widetilde{K}}$. Then G admits a weak $\mathcal{Z}$-structure.*

Remark 2. For finite $\mathrm{K}(G,1)$ complexes K and L, or more generally, compact aspherical ANRs with $\pi_1(K) \cong G \cong \pi_1(L)$, there is a G-equivariant proper homotopy equivalence $\widetilde{f} : \widetilde{K} \to \widetilde{L}$. If $j \in G$ satisfies the hypothesis of Theorem 1.3, then so does $\widetilde{f} \circ j$. Hence, the existence of such a j can be viewed as a property of G, itself.

Example 2. *For a closed, orientable, aspherical n-manifold M^n with $\widetilde{M^n} \cong \mathbb{R}^n$ (e.g., M^n a Riemannian manifold of nonpositive sectional curvature) every element of $\pi_1(M^n)$ satisfies the hypothesis of Theorem 1.2. On the other hand, for finitely generated free groups, no elements do. Of course, weak $\mathcal{Z}$-structures for both of these classes of groups are known for other reasons.*

Corollary 1.4. *If H is type F, then $H \times \mathbb{Z}$ admits a weak $\mathcal{Z}$-structure.*

Proof of Corollary. This corollary is immediate from Theorem 1.2. Alternatively, it may be obtained from Theorem 1.3. Let K be a finite $\mathrm{K}(H,1)$ and $H \times \mathbb{Z}$ act diagonally on $\widetilde{K} \times \mathbb{R}$. The nontrivial elements of $\mathbb{Z}$ satisfy the hypotheses of that theorem. $\square$

Theorems 1.2 and 1.3 are consequences of a pair of more general results, with hypotheses more topological than group-theoretic.

Theorem 1.5. *Suppose G admits a finite $K(G,1)$ complex K with the property that $\widetilde{K}$ is proper homotopy equivalent to a product $X \times Y$ of noncompact ANRs, then G admits a weak $\mathcal{Z}$-structure.*

Theorem 1.6. *Suppose G admits a finite $K(G,1)$ complex K for which $\widetilde{K}$ is proper homotopy equivalent to an ANR X that admits a proper $\mathbb{Z}$-action generated by a homeomorphism $h: X \to X$ that is properly homotopic to id_X. Then G admits a weak $\mathcal{Z}$-structure.*

Note that neither the product structure in Theorem 1.5 nor the $\mathbb{Z}$-action in Theorem 1.6 are required to have any relationship to the G-action on $\widetilde{K}$. The greater generality of these theorems is illustrated by the following improvement on Corollary 1.4.

Corollary 1.7. *If H is type F, then any semidirect product $H \rtimes_\phi \mathbb{Z}$ admits a weak $\mathcal{Z}$-structure.*

Sketch of proof. If K is a finite $\mathrm{K}(H,1)$ complex, there exists a finite $\mathrm{K}(H \rtimes_\phi \mathbb{Z}, 1)$ complex, T with universal cover $\widetilde{T}$ proper homotopy equivalent to $\widetilde{K} \times \mathbb{R}$. The corollary then follows from either Theorem 1.5 or 1.6.

To obtain T, let $f : (K,*) \to (K,*)$ be a cellular map inducing the automorphism ϕ, and let T be the mapping torus of g. The infinite-cyclic cover $\widehat{T}$ of T corresponding to $H \leq H \rtimes_\phi \mathbb{Z}$ may be viewed as a bi-infinite "mapping telescope" made up of copies of the mapping cylinder of f. The homotopy equivalence f, together with a homotopy inverse g, can be used to inductively build a proper homotopy equivalence $u : K \times \mathbb{R} \to \widehat{T}$, which may be lifted to a proper homotopy equivalence $\widetilde{u} : \widetilde{K} \times \mathbb{R} \to \widetilde{T}$. (Details provided in §3.3.) $\square$

A third variety of existence theorem for $\mathcal{Z}$-structure has, as its primary hypothesis, a condition on the end behavior of G.

Theorem 1.8. *If G is type F, 1-ended, and has pro-monomorphic fundamental group at infinity, then G admits a weak $\mathcal{Z}$-structure.*

Corollary 1.9. *If a type F group G is simply connected at infinity, then G admits a weak $\mathcal{Z}$-structure.*

Results found in [Ja], [Mi], [Pr], and [CM] show that simple connectivity at infinity is a common property for certain types of group extensions. By applying those results, some interesting overlap can be seen in the collections of groups covered by Corollary 1.9 and those covered by Theorems 1.2 and 1.3.

In the next section, we introduce some terminology and review a number of established results that are fundamental to our later arguments. In §3 we prove a variety topological theorems related to end properties of ANRs, complexes, and Hilbert cube manifolds. Most importantly, we prove $\mathcal{Z}$-compactifiability for a variety of spaces. Several results obtained there are more general than required for the group-theoretic applications in this paper, and may be of independent interest. In §4, we assemble the pieces to obtain proofs of the theorems stated above.

2. Terminology and background

2.1. **Inverse sequences of groups.** Throughout this subsection all arrows denote homomorphisms, while arrows of the type $\twoheadrightarrow$ or $\twoheadleftarrow$ denote surjections and arrows of the type $\rightarrowtail$ and $\leftarrowtail$ denote injections.

Let

$$G_0 \xleftarrow{\lambda_1} G_1 \xleftarrow{\lambda_2} G_2 \xleftarrow{\lambda_3} \cdots$$

be an inverse sequence of groups. A *subsequence* of $\{G_i, \lambda_i\}$ is an inverse sequence of the form

$$G_{i_0} \xleftarrow{\lambda_{i_0+1} \circ \cdots \circ \lambda_{i_1}} G_{i_1} \xleftarrow{\lambda_{i_1+1} \circ \cdots \circ \lambda_{i_2}} G_{i_2} \xleftarrow{\lambda_{i_2+1} \circ \cdots \circ \lambda_{i_3}} \cdots .$$

In the future we denote a composition $\lambda_i \circ \cdots \circ \lambda_j$ $(i \leq j)$ by $\lambda_{i,j}$.

Sequences $\{G_i, \lambda_i\}$ and $\{H_i, \mu_i\}$ are *pro-isomorphic* if, after passing to subsequences, there exists a commuting "ladder diagram":

(2.1)
$$\begin{tikzcd}
G_{i_0} & & G_{i_1} \arrow[ll, "\lambda_{i_0+1,i_1}"'] \arrow[dl] & & G_{i_2} \arrow[ll, "\lambda_{i_1+1,i_2}"'] \arrow[dl] & & G_{i_3} \arrow[ll, "\lambda_{i_2+1,i_3}"'] \arrow[dl] & \cdots \\
& H_{j_0} \arrow[ul] & & H_{j_1} \arrow[ll, "\mu_{j_0+1,j_1}"] \arrow[ul] & & H_{j_2} \arrow[ll, "\mu_{j_1+1,j_2}"] \arrow[ul] & & \cdots \arrow[ll, "\mu_{j_2+1,j_3}"]
\end{tikzcd}$$

Clearly an inverse sequence is pro-isomorphic to any of its subsequences. To avoid tedious notation, we sometimes do not distinguish $\{G_i, \lambda_i\}$ from its subsequences. Instead we assume that $\{G_i, \lambda_i\}$ has the desired properties of a preferred subsequence—prefaced by the words "after passing to a subsequence and relabeling".

An inverse sequence $\{G_i, \lambda_i\}$ is called *pro-monomorphic* if it is pro-isomorphic to an inverse sequence of monomorphisms and *pro-epimorphic* (more commonly called *semistable* or *Mittag-Leffler*) if it is pro-isomorphic to an inverse sequence of epimorphisms. It is *stable* if it is pro-isomorphic to a constant inverse sequence $\{H, \mathrm{id}_H\}$, or equivalently, to an inverse sequence of isomorphisms. A sequence is stable if and only if it is both pro-monomorphic and pro-epimorphic.

A few more special classes of inverse sequences will be of special interest in this paper. A sequence that is pro-isomorphic to the trivial sequence $1 \leftarrow 1 \leftarrow 1 \leftarrow \cdots$ is called *pro-trivial*; a sequence pro-isomorphic to an inverse sequence of finitely generated groups is called *pro-finitely generated*; and a sequence that is pro-isomorphic to an inverse sequence of free groups is called *pro-free*. A sequence that is both pro-finitely generated and pro-free is easily seen to be pro-isomorphic to an inverse sequence of finitely generated free groups. We call such a sequence *pro-finitely generated free*.

The *inverse limit* of a sequence $\{G_i, \lambda_i\}$ is the subgroup of $\prod G_i$ defined by

$$\varprojlim \{G_i, \lambda_i\} = \left\{ (g_0, g_1, g_2, \cdots) \in \prod_{i=0}^{\infty} G_i \,\middle|\, \lambda_i(g_i) = g_{i-1} \right\}.$$

In the special case where $\{G_i, \lambda_i\}$ is an inverse sequence of abelian groups, we also define the *derived limit*[2] to be the following quotient group:

$$\varprojlim{}^1 \{G_i, \lambda_i\} = \left(\prod_{i=0}^{\infty} G_i \right) / \left\{ (g_0 - \lambda_1 g_1, g_1 - \lambda_2 g_2, g_2 - \lambda_3 g_3, \cdots) \mid \; g_i \in G_i \right\}$$

It is a standard fact that pro-isomorphic inverse sequences of groups have isomorphic inverse limits and, pro-isomorphic inverse sequences of abelian groups have isomorphic derived limits.

2.2. **Absolute neighborhood retracts.** Throughout this paper, all spaces are assumed to be separable metric. A locally compact space X is an ANR (*absolute neighborhood retract*) if it can be embedded into $\mathbb{R}^n$ or, if necessary, $\mathbb{R}^\infty$ (a countable product of real lines) as a closed set in such a way that there exists a retraction $r : U \to X$, where U is a neighborhood of X. If the entire space $\mathbb{R}^n$ or $\mathbb{R}^\infty$ retracts onto X, we call X an AR (absolute retract). If X is finite-dimensional, all mention of $\mathbb{R}^\infty$ can be omitted. A finite-dimensional ANR is called an ENR (*Euclidean neighborhood retract*) and a finite-dimensional AR an ER. With a little effort it can be shown that an AR [resp., ER] is simply a contractible ANR [resp., ENR].

A space X is *locally contractible* if every ball $B(x;\varepsilon)$ contains a ball $B(x;\delta)$ that contracts within $B(x;\varepsilon)$. It is a standard fact that every ANR is locally contractible. For finite-dimensional spaces, that property characterizes ANRs. In other words, a locally compact, finite-dimensional space X is an ANR (and hence an ENR) if and only if it is locally contractible. It follows that every finite-dimensional locally finite polyhedron or CW complex is an ENR; if it is contractible, it is an ER.

The following famous result will be used in this paper.

Theorem 2.1 (West, [We]). *Every ANR is homotopy equivalent to a locally finite polyhedron. Every compact ANR is homotopy equivalent to a finite polyhedron.*

[2] The definition of derived limit can be generalized to include nonableian groups (see [Ge2, §11.3]), but that will not be needed in this paper.

2.3. **Proper maps and homotopies.** When working with noncompact space, the notion of 'properness' is crucial. A map $f : X \to Y$ is *proper* if $f^{-1}(C)$ is compact whenever $C \subseteq Y$ is compact. Maps $f_0, f_1 : X \to Y$ are *properly homotopic*, denoted $f_0 \overset{p}{\simeq} f_1$ if there exists a proper map $H : X \times [0,1] \to Y$ with $H_0 = f_0$ and $H_1 = f_1$. Spaces X and Y are *proper homotopy equivalent*, denoted $X \overset{p}{\simeq} Y$, if there exist proper maps $f : X \to Y$ and $g : Y \to X$ with $gf \overset{p}{\simeq} \mathrm{id}_X$ and $fg \overset{p}{\simeq} \mathrm{id}_Y$.

2.4. **Ends of spaces and the fundamental group at infinity.** A subset N of a space X is a *neighborhood of infinity* if $\overline{X - N}$ is compact. A standard argument shows that, when X is an ANR and C is a compact subset of X, $X - C$ has at most finitely many unbounded components, i.e., finitely many components with noncompact closures. If $X - C$ has both bounded and unbounded components, the situation can be simplified by letting C' consist of C together with all bounded components. Then C' is compact, and $X - C'$ consists entirely of unbounded components.

We say that X *has k ends* if there exists a compactum $C \subseteq X$ such that, for every compactum D with $C \subset D$, $X - D$ has exactly k unbounded components. When k exists, it is uniquely determined; if k does not exist, we say X has *infinitely many ends*. Thus, a space is 0-ended if and only if X is compact, and 1-ended if and only if it contains arbitrarily small connected neighborhoods of infinity.

A nested sequence $N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$ of neighborhoods of infinity, with each $N_i \subseteq \operatorname{int} N_{i-1}$, is *cofinal* if $\bigcap_{i=0}^{\infty} N_i = \varnothing$. Such a sequence is easily obtained: choose an exhaustion of X by compacta $C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$, with $C_{i-1} \subseteq \operatorname{int} C_i$; then let $N_i = X - C_i$. When closed neighborhoods of infinity are required, let $N_i - \overline{X - C_i}$.

Given a nested cofinal sequence $\{N_i\}_{i=0}^{\infty}$ of neighborhoods of infinity, base points $p_i \in N_i$, and paths $r_i \subset N_i$ connecting p_i to p_{i+1}, we obtain an inverse sequence:

$$\pi_1(N_0, p_0) \xleftarrow{\lambda_1} \pi_1(N_1, p_1) \xleftarrow{\lambda_2} \pi_1(N_2, p_2) \xleftarrow{\lambda_3} \cdots . \tag{2.2}$$

Here, each $\lambda_{i+1} : \pi_1(N_{i+1}, p_{i+1}) \to \pi_1(N_i, p_i)$ is the homomorphism induced by inclusion followed by the change of base point isomorphism determined by r_i. The proper ray $r : [0, \infty) \to X$ obtained by piecing together the r_i in the obvious manner is referred to as the *base ray* for the inverse sequence, and the pro-isomorphism class of the inverse sequence is called the *fundamental group at infinity of X based at r* and is denoted pro-$\pi_1(\varepsilon(X), r)$. It is a standard fact that pro-$\pi_1(X, r)$ is independent of the sequence of neighborhoods $\{N_i\}$ or the base points—provided those base points tend to infinity along the ray r, and corresponding subpaths of r are used in defining the λ_i. More generally, pro-$\pi_1(\varepsilon(X), r)$ depends only upon the proper homotopy class of r. If X is 1-ended and pro-$\pi_1(\varepsilon(X), r)$ is semistable for some proper ray r, it can be shown that all proper rays in X are properly homotopic; in that case we say that X is *strongly connected at infinity*. When X is strongly connected at infinity, it is safe to omit mention of the base ray and to speak generally of the *fundamental group at infinity of X*, and denote it by pro-$\pi_1(\varepsilon(X))$. If X is 1-ended and pro-$\pi_1(\varepsilon(X), r)$ is pro-trivial, we call X *simply connected at infinity*.

The fundamental group at infinity is clearly not a homotopy invariant of a space, but it is a proper homotopy invariant. More precisely, if $f : X \to Y$ is a proper homotopy equivalence, then pro-$\pi_1(\varepsilon(X), r)$ is pro-isomorphic to pro-$\pi_1(\varepsilon(Y), f \circ r)$.

For a group G of type F, the universal cover $\widetilde{K}$ of a finite $\mathrm{K}(G,1)$ complex K is well-defined up to proper homotopy type. So the number of ends of G is well-defined; and if $\widetilde{K}$ is 1-ended, except for the issue of a base ray, we may view pro-$\pi_1\left(\varepsilon(\widetilde{K}), r\right)$ as an invariant of G. The base ray issue goes away when pro-$\pi_1\left(\varepsilon(\widetilde{K}), r\right)$ is semistable, so there is no ambiguity in defining a 1-ended G to have semistable, stable, or trivial fundamental group at infinity, according to whether pro-$\pi_1\left(\varepsilon(\widetilde{K}), r\right)$ has the corresponding property. With some additional work, it can be shown that the property of pro-$\pi_1\left(\varepsilon(\widetilde{K}), r\right)$ being pro-monomorphic is also independent of base ray and, thus, attributable to G.

Although not needed for this paper, the requirement in the previous paragraph, that G have type F can be significantly weakened. In particular, if G is finitely presented, and L is any finite complex with fundamental group G, then the number of ends of $\widetilde{L}$ and the properties of pro-$\pi_1\left(\varepsilon(\widetilde{L}), r\right)$ discussed above, are invariants of G. Thus, for example, a finitely presented group G is called *simply connected at infinity* if $\widetilde{L}$ has that property. For more information about the fundamental group at infinity of spaces and groups, including proofs of the made in this section, see [Ge2] or [Gu3].

2.5. **Finite domination and inward tameness.** A space Y has *finite homotopy type* if it is homotopy equivalent to a finite CW complex; it is *finitely dominated* if there is a finite complex K and maps $u : Y \to K$ and $d : K \to Y$ such that $d \circ u \simeq \mathrm{id}_Y$. If Y is an ANR, then Y is finitely dominated if and only if there exists a self-homotopy that 'pulls Y into a compact subset', i.e., $H : Y \times [0,1] \to Y$ such that $H_0 = \mathrm{id}_Y$ and $\overline{H_1(Y)}$ is compact. This equivalence is easily verified when (for example) K is a locally finite polyhedron; a discussion of the general case can be found in [Gu3, §3.4].

The following clever observation will be used later.

Theorem 2.2 (Mather, [Ma])**.** *If a space Y is finitely dominated, then $Y \times \mathbb{S}^1$ has finite homotopy type.*

An ANR X is *inward tame* if, for every closed neighborhood of infinity N in X, there is a homotopy $K : N \times [0,1] \to N$ with $K_0 = \mathrm{id}_N$ and $\overline{K_1(N)}$ compact (a homotopy pulling N into a compact subset). By an easy application of Borsuk's Homotopy Extension Property, this is equivalent to the existence of a cofinal sequence $\{N_i\}$ of closed neighborhoods of infinity, each of which can be pulled into a compact set. If X contains a cofinal sequence $\{N_i\}$ of closed ANR neighborhoods[3] of infinity, then inward tameness is equivalent to each of those (hence, all closed ANR neighborhoods of infinity) being finitely dominated. [Gu3, §3.5] provides additional details.

[3]In this case, X is called *sharp at infinity.* Most commonly arising ANRs, for example: locally finite polyhedra, manifolds, proper CAT(0) spaces, and Hilbert cube manifolds) are sharp at infinity.

Inward tameness is an invariant of proper homotopy type. Roughly speaking, if $f : X \to Y$ and $g : Y \to X$ are proper homotopy inverses and H is a homotopy that pulls a neighborhood of infinity of X into a compact set, then $f \circ H_t \circ g$ pulls a neighborhood of Y into a compact set. More details can be found in [Gu3, §3.5].

2.6. **Some basic K-theory.** An important result from [Wa] asserts that, for each finitely dominated, connected space Y, there is a well-defined obstruction $\sigma(Y)$, lying in the *reduced projective class group* $\widetilde{K}_0(\mathbb{Z}[\pi_1(Y)])$, which vanishes if and only if Y has finite homotopy type.

A related algebraic construction is the *Whitehead group.* If (A, B) is a pair of connected, finite CW complexes and $B \hookrightarrow A$ is a homotopy equivalence, then there is a well-defined obstruction $\tau(B)$, lying in an abelian group $\mathrm{Wh}(\pi_1(B))$ that vanishes if and only if $B \hookrightarrow A$ is a simple homotopy equivalence. Definitions and details can be found in [Co].

Both of the above algebraic constructs act as functors in the sense that, if $\lambda : G \to H$ is a group homomorphism, there are naturally induced homomorphims $\lambda_* : \widetilde{K}_0(\mathbb{Z}[G]) \to \widetilde{K}_0(\mathbb{Z}[G])$ and $\lambda_* : \mathrm{Wh}(G) \to \mathrm{Wh}(H)$.

For the purposes of this paper, the main thing we need to know about $\widetilde{K}_0(\mathbb{Z}[\pi_1(Y)])$ or $\mathrm{Wh}(\pi_1(B))$ is contained in a famous result of Bass-Heller-Swan [BHS].

Theorem 2.3. *If G is a finitely generated free group, then both $\widetilde{K}_0(\mathbb{Z}[G])$ and $\mathrm{Wh}(G)$ are the trivial group.*

2.7. **Mapping cylinders, mapping tori, and mapping telescopes.** For any map $f : K \to L$ and closed interval $[a, b]$, the *mapping cylinder* $\mathcal{M}_{[a,b]}(f)$ is the quotient space $L \sqcup (K \times [a, b]) / \sim$, where $\sim$ is the equivalence relation generated by the rule $(x, a) \sim f(x)$ for all $x \in K$. Let $q_{[a,b]} : L \sqcup (K \times [a, b]) \to \mathcal{M}_{[a,b]}(f)$ be the quotient map. Then, for each $r \in (a, b]$, $q_{[a,b]}$ restricts to an embedding of $K \times \{r\}$ into $\mathcal{M}_{[a,b]}(f)$; denote the image of $K \times \{r\}$ by K_r. The quotient map is also an embedding when restricted to L; let $L_a \subseteq \mathcal{M}_{[a,b]}(f)$ be that copy of L. We call K_b the *domain end* and L_a the *range end* of $\mathcal{M}_{[a,b]}(f)$. Note the existence of a projection map $p_{[a,b]} : \mathcal{M}_{[a,b]}(f) \to [a, b]$ for which $p_{[a,b]}^{-1}(r) = K_r$ is a copy of K for each $r \in (a, b]$ and $p_{[a,b]}^{-1}(a) = L_a$ is a copy of L. Note also that, when $K = L$, i.e., f maps K to itself, all of the above still applies. In that case, each point preimage of $p_{[a,b]}$ is a copy of K, but the copy K_a is different from the others, in that it is not necessarily parallel to neighboring copies.

Remark 3. Clearly the topological type of $\mathcal{M}_{[a,b]}(f)$ does not depend on the interval $[a, b]$, and for most purposes can be taken to be $[0, 1]$. But in the treatment that follows, it will be useful to allow the interval to vary.

The following standard application of mapping cylinders will be used several times in this paper. A proof, in which properness is not mentioned, can be found in [Du, p.372]. For our purposes, it is only the easy (converse) direction of the proper assertion that will be used.

Lemma 2.4. *A map $f : K \to L$ between ANRs is a homotopy equivalence if and only if there exists a strong deformation retraction of $\mathcal{M}_{[a,b]}(f)$ onto M_b. It is a proper homotopy equivalence if and only if there exists a proper strong deformation retraction of $\mathcal{M}_{[a,b]}(f)$ onto K_b.*

The *bi-infinite mapping telescope* of a map $f : K \to K$ is obtained by gluing together infinitely many mapping cylinders. More precisely,

$$\mathrm{Tel}_f(K) = \cdots \cup \mathcal{M}_{[-2,-1]}(f) \cup \mathcal{M}_{[-1,0]}(f) \cup \mathcal{M}_{[0,1]}(f) \cup \mathcal{M}_{[1,2]}(f) \cup \mathcal{M}_{[2,3]}(f) \cup \cdots$$

where the gluing is accomplished by identifying the domain end of each $\mathcal{M}_{[n-1,n]}(f)$ with the range end of $\mathcal{M}_{[n,n+1]}(f)$. Notationally, this works well since, under the convention described above, each is denoted K_n. Projection maps may be pieced together to obtain a projection $p : \mathrm{Tel}_f(K) \to \mathbb{R}$, for which $p^{-1}(r) = K_r$ is a copy of K, for each $r \in \mathbb{R}$. A schematic of $\mathrm{Tel}_f(K)$ is contained in Figure 1 of §3.3.

The *mapping torus* of $f : K \to K$ is obtained from $\mathcal{M}_{[0,1]}(f)$ by identifying K_0 with K_1. It may also be defined more directly as the quotient space

$$\mathrm{Tor}_f(K) = K \times [0,1] / \sim$$

where $\sim$ is the equivalence relation generated by $(x,0) \sim (f(x),1)$ for each $x \in K$. The following facts about mapping mapping tori are standard.

Lemma 2.5. *Let K be a connected ANR, $f : (K,p) \to (K,q)$ a map that induces an isomorphism $\varphi : \pi_1(K,p) \to \pi_1(K,q)$, and λ a path in K from q to p. Then*

1. *$\pi_1(\mathrm{Tor}_f(K), (p,0)) \cong \pi_1(K,p) \rtimes_\varphi \langle t \rangle$, where t is an infinite order element represented by the loop $(\{p\} \times [0,1]) \cdot \lambda$, and*
2. *the infinite cyclic cover of $\mathrm{Tor}_f(K)$ corresponding to the projection $\pi_1(K,p) \rtimes_\varphi \langle t \rangle \to \langle t \rangle$ is the bi-infinite mapping telescope $\mathrm{Tel}_f(K)$.*

The following fact about mapping tori can be found in [GG], where it plays a crucial role. We will make significant use of it here as well.

Lemma 2.6. *Let X be a connected ANR that admits a proper $\mathbb{Z}$-action generated by a homeomorphism $j : X \to X$. Then $(\langle j \rangle \backslash X) \times \mathbb{R}$ is homeomorphic to $\mathrm{Tor}_j(X)$.*

2.8. **Hilbert cube manifolds.** The *Hilbert cube* is the infinite product

$$\mathcal{Q} = \prod_{i=1}^{\infty} [-1,1], \text{ with metric } d((x_i),(y_i)) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

A *Hilbert cube manifold* is a space X with the property that each $x \in X$ has a neighborhood homeomorphic to $\mathcal{Q}$. Although we are primarily interested in finite-dimensional spaces, Hilbert cube manifolds play a key role in this paper. A pair of classical results will allow us to move between the categories of ANRs and locally finite polyhedra.

Theorem 2.7 (Edwards, [Ed])**.** *If A is an ANR, then $A \times \mathcal{Q}$ is a Hilbert cube manifold.*

Theorem 2.8 (Chapman, [Ch])**.** *If X is a Hilbert cube manifold, then there is a locally finite polyhedron K for which $X \approx K \times \mathcal{Q}$.*

2.9. **$\mathcal{Z}$-sets and $\mathcal{Z}$-compactifications.** A closed subset A of an ANR Y is a *$\mathcal{Z}$-set* if either of the following equivalent conditions is satisfied:

- There exists a homotopy $H : Y \times [0,1] \to Y$ such that $H_0 = \mathrm{id}_Y$ and $H_t(X) \subseteq Y - A$ for all $t > 0$. (We say that H *instantly homotopes* Y off from A.)
- For every open set U in Y, $U - A \hookrightarrow U$ is a homotopy equivalence.

A *$\mathcal{Z}$-compactification* of a space X is a compactification $\overline{X} = X \sqcup Z$ with the property that Z is a $\mathcal{Z}$-set in $\overline{X}$. In this case, Z is called a *$\mathcal{Z}$-boundary* for X. Implicit in this definition is the requirement that $\overline{X}$ be an ANR; moreover, since an open subset of an ANR is an ANR, X itself must be an ANR to be a candidate for $\mathcal{Z}$-compactification. Hanner's Theorem [Ha] ensures that every compactification $\overline{X}$ of an ANR X, for which $\overline{X} - X$ satisfies either of the "negligibility conditions" in the definition of $\mathcal{Z}$-set, is necessarily an ANR; hence, it is a $\mathcal{Z}$-compactification. By a similar result in dimension theory, $\mathcal{Z}$-compactification does not increase dimension; so, if X is an ENR, so is $\overline{X}$.

Example 3. *The compactification of $\mathbb{R}^n$ obtained by adding the $(n-1)$-sphere at infinity is the prototypical $\mathcal{Z}$-compactification. More generally, addition of the visual boundary to a proper CAT(0) space is a $\mathcal{Z}$-compactification. In* [BM]*, it is shown that, for a torsion-free δ-hyperbolic group G, an appropriately chosen Rips complex can be $\mathcal{Z}$-compactified by adding the Gromov boundary ∂G.*

The purely topological question of when an ANR or ENR admits a $\mathcal{Z}$-compactification is an open question. However, Chapman and Siebenmann [CS] have given a complete classification of $\mathcal{Z}$-compactifiability for Hilbert cube manifolds. That result, in combination with Theorem 2.7, has substantial implications for the general case.

Here we provide a slightly simplified version of the Chapman-Siebenmann theorem. First, we state the result only for 1-ended Hilbert cube manifolds X, since that is all we need in this paper. Second, we simplify the definitions of $\sigma_\infty(X)$ and $\tau_\infty(X)$ by beginning with a prechosen nested, cofinal sequence of nice neighborhoods of infinity. It is true, but would take some time, to explain why the resulting obstructions do not depend on that choice.

A particularly nice variety of closed neighborhood of infinity $N \subseteq X$ is one that is, itself, a Hilbert cube manifold and whose topological boundary $\mathrm{Bd}_X N$ is a Hilbert cube manifold with a neighborhood in X homeomorphic to $\mathrm{Bd}_X N \times [-1,1]$. Call such neighborhoods of infinity *clean*. By applying Theorems 2.7 and 2.8, clean neighborhoods of infinity are easily found.

Theorem 2.9 (Chapman-Siebenmann)**.** *Let X be a 1-ended Hilbert cube manifold and $\{N_i\}$ a nested cofinal sequence of connected clean neighborhoods of infinity. Then X admits a $\mathcal{Z}$-compactification if and only if each of the following conditions holds:*

a) *X is inward tame.*
b) *$\sigma_\infty(X) \in \varprojlim \left\{ \widetilde{K}_0(\mathbb{Z}\pi_1(N_i)), \lambda_{i*} \right\}$ is zero.*
c) *$\tau_\infty(X) \in \varprojlim^1 \{\mathrm{Wh}(\pi_1(N_i)), \lambda_{i*}\}$ is zero.*

The inverse sequences $\left\{\widetilde{K}_0(\mathbb{Z}\pi_1(N_i)),\lambda_{i*}\right\}$ and $\{\mathrm{Wh}(\pi_1(N_i)),\lambda_{i*}\}$ in conditions b) and c) are obtained by applying the $\widetilde{K}_0$-functor and the Wh-functor to sequence (2.2) on page 7. The obstruction $\sigma_\infty(X)$ is just the sequence $(\sigma(N_0),\sigma(N_1),\sigma(N_2),\cdots)$ of Wall finiteness obstructions of the N_i. Condition a) ensures that each N_i is finitely dominated, so those obstructions are all defined; without condition a), there is no condition b). Similarly, condition c) requires condition b). It is related to the Whitehead torsion of inclusions $\mathrm{Bd}_X N_i \hookrightarrow \overline{N_i - N_{i+1}}$, after the N_i have been improved significantly so that those inclusions are homotopy equivalences. The reader should consult [CS] for details or [Gu3, §8.2] for a less formal discussion of Theorem 2.9. For our purposes, those details are not so important since the obstructions arising here will be shown to vanish by straightforward applications of Theorem 2.3.

Remark 4. Condition a) makes sense for an arbitrary ANR X. If that X satisfies a) and is sharp at infinity, then condition b) also makes immediate sense; it is satisfied if and only if X contains arbitrarily small closed ANR neighborhoods of infinity having finite homotopy type. Condition c) is more problematic; even if a) and b) are satisfied, if X is not a Hilbert cube manifold, it may be impossible to find neighborhoods of infinity N_i with each $\mathrm{Bd}_X N_i \hookrightarrow \overline{N_i - N_{i+1}}$ a homotopy equivalence—an example can be found in [GT]. The solution to this problem is to *define* the obstructions for an ANR X to be the corresponding obstructions for the Hilbert cube manifold $X \times \mathcal{Q}$. Then a)-c) are necessary for $\mathcal{Z}$-compactifiability of X; unfortunately, they are not sufficient. [Gu1] exhibits a locally finite 2-dimensional polyhedron K that satisfies a)-c), but is not $\mathcal{Z}$-compactifiable. A suitable characterization of $\mathcal{Z}$-compactifiable ANRs (or even locally finite polyhedra) is an open question.

For an ANR X, Theorem 2.9 allows us to determine whether $X \times \mathcal{Q}$ is $\mathcal{Z}$-compactifiable. The following result, with $\mathbb{I} = [-1,1]$, frequently allows us to restore finite-dimensionality.

Theorem 2.10 (Ferry, [Fe])**.** *If K is a finite-dimensional locally finite polyhedron and $K \times \mathcal{Q}$ is $\mathcal{Z}$-compactifiable, then $K \times \mathbb{I}^{2\cdot\dim K+5}$ is $\mathcal{Z}$-compactifiable.*

3. Topological results

In this section we prove a variety of topological results that are the primary ingredients in the proofs of our main theorems. We have broken the section into four parts: the first contains results about products of spaces; the second deals with spaces that admit a proper $\mathbb{Z}$-action generated by a homeomorphism properly homotopic to the identity; the third looks at mapping tori of self-homotopy equivalences; and the final subsection looks at spaces that are simply connected at infinity.

3.1. **Products of noncompact spaces.**

Lemma 3.1. *Let X be an ANR that is finitely dominated. Then $X \times \mathbb{R}$ is inward tame.*

Proof. Since inward tameness is an invariant of proper homotopy type, we may use Theorem 2.1 to reduce to the case that X is a locally finite polyhedron. For that case, the proof given in [Gu2, Prop. 3.1] for open manifolds is valid, with only minor modifications. With a few additional modifications, the appeal to Theorem 2.1 can be eliminated. □

The next lemma requires a new definition. We say that an ANR X is *movably finitely dominated* if, for every neighborhood of infinity $N \subseteq X$, there is a self-homotopy of X that pulls X into a compact subset of N, i.e., $H : X \times [0,1] \to X$ such that $H_0 = \mathrm{id}_X$ and $\overline{H_1(X)}$ is compact and contained in N. The motivation for this definition becomes immediately clear in the following lemma. The most important examples are the simplest—every contractible ANR is movably finitely dominated, since it is dominated by each singleton subset.

Lemma 3.2. *Let X and Y be connected, noncompact, movably finitely dominated ANRs. Then $X \times Y$ is inward tame.*

Proof. Let $A \subseteq X$ and $B \subseteq Y$ be compact and $N = \overline{(X \times Y) - (A \times B)}$ the corresponding closed neighborhood of infinity. It suffices to prove:

Claim. *There exits a homotopy $J : N \times [0,1] \to N$ with $J_0 = \mathrm{id}_N$ and $\overline{J_1(N)}$ compact.*

Choose compacta $A' \subseteq X$ and $B' \subseteq Y$ such that $A \subseteq \mathrm{int}_X A'$ and $B \subseteq \mathrm{int}_X B'$, and let $\lambda : X \to [0,1]$ and $\mu : Y \to [0,1]$ be Urysohn functions with $\lambda(A) = 0 = \mu(B)$ and $\lambda\left(\overline{X - A'}\right) = 1 = \mu\left(\overline{Y - B'}\right)$. Then choose compact $K \subseteq X - A'$ and $L \subseteq Y - B'$ along with homotopies $F : X \times [0,1] \to X$ such that $F_0 = \mathrm{id}_X$ and $F_1(X) \subseteq K$ and $G : Y \times [0,1] \to Y$ such that $G_0 = \mathrm{id}_Y$ and $G_1(X) \subseteq L$.

We will build a homotopy H that pulls $X \times Y$ into a compact subset while fixing $A \times B$. By arranging that tracks of points from N stay in N, the restriction of this homotopy will satisfy the claim.

Define $\widehat{F} : X \times Y \times [0,1] \to X \times Y$ by $\widehat{F}(x,y,t) = (F(x, \mu(y) \cdot t), y)$ and note that:

- $\widehat{F}_1(X \times Y) \subseteq (X \times B') \cup (K \times Y)$,
- $\widehat{F}_t\Big|_{X \times B} = \mathrm{id}$ for all t, and
- tracks of points in N stay in N.

Similarly, let $\widehat{G} : X \times Y \times [0,1] \to X \times Y$ by $\widehat{G}(x,y,t) = (x, G(y, \lambda(x) \cdot t))$ and note that:

- $\widehat{G}_1(X \times Y) \subseteq (A' \times Y) \cup (X \times L)$,
- $\widehat{G}\Big|_{A \times Y} = \mathrm{id}$, and
- tracks of points in N stay in N.

Now define $H : X \times Y \times [0,1] \to X \times Y$ by the rule.

$$H_t = \begin{cases} \widehat{F}_{3t} & 0 \leq t \leq \frac{1}{3} \\ \widehat{G}_{3t-1} \circ \widehat{F}_1 & \frac{1}{3} \leq t \leq \frac{2}{3} \\ \widehat{F}_{3t-2} \circ \widehat{G}_1 \circ \widehat{F}_1 & \frac{2}{3} \leq t \leq 1 \end{cases}$$

A quick check shows that $H_1 (X \times Y)$ is contained in the compact set $\widehat{F}_1 \widehat{G}_1 (A' \times B') \cup (K \times L)$; moreover, since the tracks of H are all concatenations of tracks of $\widehat{F}$ and $\widehat{G}$, $A \times B$ stays fixed and tracks of points from N stay in N. Letting J be the restriction of H completes the claim. □

Corollary 3.3. *The product of any two noncompact ARs is inward tame.*

Lemma 3.4. *Let X and Y be noncompact, simply connected ANRs. Then $X \times Y$ contains arbitrarily small path-connected neighborhoods of infinity, each with a fundamental group that is finitely generated and free.*

Proof. Let $U \subseteq X$ and $V \subseteq Y$ be open neighborhoods of infinity, consisting of finitely many unbounded path-connected components $\{U_i\}_{i=1}^{k_1}$ and $\{V_j\}_{j=1}^{k_2}$, respectively. Then the corresponding rectangular neighborhood of infinity $R = (U \times Y) \cup (X \times V)$ may be covered by the finite collection of path-connected open sets $\{U_i \times Y\}_{i=1}^{k_1} \cup \{X \times V_j\}_{j=1}^{k_2}$ in which each of the two subcollections is pairwise disjoint, and each $U_i \times Y$ intersects each $X \times V_j$ in the path-connected set $U_i \times V_j$. The nerve of this cover is the complete bipartite graph K_{k_1,k_2} and the connectedness of this graph implies the path-connectedness of R. A straight-forward application of the Generalized van Kampen Theorem to the corresponding generalized graph of groups (see [Ge2, Th.6.2.11]) shows that the fundamental group of R is free on $(k_1 - 1)(k_2 - 1)$ generators, the key observation being that each element of a vertex group $\pi_1 (U_i \times Y)$ can be represented by a loop in $U_i \times V_j$ which then contracts in $X \times V_j$, and similarly for elements of vertex groups $\pi_1 (X \times V_j)$. □

Theorem 3.5. *Let X and Y be noncompact, simply connected, movably finitely dominated Hilbert cube manifolds. Then $X \times Y$ is $\mathcal{Z}$-compactifiable.*

Proof. By a combination of Corollary 3.3, Lemma 3.4, and Theorem 2.3, $X \times Y$ satisfies all conditions of Theorem 2.9. □

Theorem 3.6. *Let P_1 and P_2 be noncompact, simply connected, moveably finitely dominated, finite-dimensional, locally finite polyhedra. Then $P_1 \times P_2 \times \mathbb{I}^{2(\dim P_1 + \dim P_2)+5}$ is $\mathcal{Z}$-compactifiable.*

Proof. Apply Theorems 2.7 and 3.5 to $P_1 \times P_2 \times \mathcal{Q}$; then use Theorem 2.10. □

3.2. **Spaces admitting homotopically simple $\mathbb{Z}$-actions.** In this section we consider spaces X that admit a proper $\mathbb{Z}$-action generated by a homeomorphism properly homotopic to id_X. Under the right circumstances, that hypothesis has significant implication for the topology of X.

Lemma 3.7. *Let X be an ANR that admits a proper $\mathbb{Z}$-action generated by a homeomorphism $j : X \to X$ that is properly homotopic to* id_X*. Then*

(1) *if the action is cocompact, X is 2-ended;*
(2) *if the action is not cocompact, X is 1-ended; and*
(3) *if X is finitely dominated, then X is inward tame.*

Proof. By Lemma 2.6, $(\langle j \rangle \backslash X) \times \mathbb{R} \approx \mathrm{Tor}_j(X)$, and since $j \overset{p}{\simeq} \mathrm{id}_X$, the latter space is proper homotopy equivalent to $X \times \mathbb{S}^1$. Now $(\langle j \rangle \backslash X) \times \mathbb{R}$ has either two or one ends, according to whether $\langle j \rangle \backslash X$ is compact or noncompact, and since the number of ends is a proper homotopy invariant, the same is true for $X \times \mathbb{S}^1$. Since $X \times \mathbb{S}^1$ has the same number of ends as X, the first two assertions follow.

Next assume that X is finitely dominated. By Theorem 2.2, $X \times \mathbb{S}^1$ has finite homotopy type, so by the above equivalences, $\langle j \rangle \backslash X$ also has finite homotopy type. By Lemma 3.1, $(\langle j \rangle \backslash X) \times \mathbb{R}$ is inward tame, and since inward tameness is an invariant of proper homotopy type, $X \times \mathbb{S}^1$ is inward tame. It follows that X is inward tame since, if N is a closed neighborhood of infinity in X, then $N \times \mathbb{S}^1$ is a closed neighborhood of infinity in $X \times \mathbb{S}^1$; and a homotopy that pulls $N \times \mathbb{S}^1$ into a compact subset projects to a homotopy that pulls N into a compact subset. □

Lemma 3.8. *Let X be a simply connected ANR that admits a proper $\mathbb{Z}$-action generated by a homeomorphism $j : X \to X$ that is properly homotopic to* id_X*. Then*

(1) *if the action is cocompact, X is simply connected at each of its two ends, and*
(2) *if the action is not cocompact, X is strongly connected at infinity and* pro-$\pi_1(\varepsilon(X))$ *is pro-finitely generated free.*

Proof. The proof is primarily an application of work done in [GM2]; we add a few observations to make those results fit our situation more precisely. For both assertions, we use the equivalences:

$$(\langle j \rangle \backslash X) \times \mathbb{R} \approx \mathrm{Tor}_j(X) \overset{p}{\simeq} X \times \mathbb{S}^1. \tag{3.1}$$

First assume that $\langle j \rangle \backslash X$ is compact. Then $(\langle j \rangle \backslash X) \times \mathbb{R}$ is 2-ended and the natural choices of base rays: $r_- = \{p\} \times (-\infty, 0]$ and $r_+ = \{p\} \times [0, \infty)$, along with the natural choice of neighborhoods of infinity $(\langle j \rangle \backslash X) \times (-\infty, -n] \cup [n, \infty)$ yield representations of pro-$\pi_1(\varepsilon((\langle j \rangle \backslash X) \times \mathbb{R}), r_\pm)$ of the form $\mathbb{Z} \xleftarrow{\mathrm{id}} \mathbb{Z} \xleftarrow{\mathrm{id}} \mathbb{Z} \xleftarrow{\mathrm{id}} \cdots$. The proper homotopy equivalence promised above does the same for the two ends of $X \times \mathbb{S}^1$. Clearly, this can happen only if X is simply connected at each of its two ends.

In the non-cocompact case, $(\langle j \rangle \backslash X) \times \mathbb{R}$ is 1-ended and by [GM2, Prop. 3.12], with an appropriate choice of base ray, pro-$\pi_1(\varepsilon((\langle j \rangle \backslash X) \times \mathbb{R}), r)$ may be represented by an inverse sequence

$$F_0 \times \langle a \rangle \overset{\lambda_1 \times \mathrm{id}}{\twoheadleftarrow} F_1 \times \langle a \rangle \overset{\lambda_2 \times \mathrm{id}}{\twoheadleftarrow} F_2 \times \langle a \rangle \overset{\lambda_3 \times \mathrm{id}}{\twoheadleftarrow} \cdots \tag{3.2}$$

where each F_i is a finitely generated free group, λ_i takes F_{i+1} onto F_i, and $\langle a \rangle$ is an infinite cyclic group corresponding to a 'copy' of $\pi_1((\langle j \rangle \backslash X) \times \{r_i\})$, for increasingly large r_i. Semistability of this sequence implies that $(\langle j \rangle \backslash X) \times \mathbb{R}$, and hence $X \times \mathbb{S}^1$, is strongly connected at infinity. This allows us to dispense with mention of base rays.

It also implies that X is strongly connected at infinity, so pro-$\pi_1(\varepsilon(X))$ is semistable and may be represented by an inverse sequence of surjections $H_0 \overset{\mu_1}{\twoheadleftarrow} H_1 \overset{\mu_2}{\twoheadleftarrow} H_2 \overset{\mu_3}{\twoheadleftarrow} \cdots$. It follows that pro-$\pi_1(\varepsilon(X \times \mathbb{S}^1))$ may be represented by

$$H_0 \times \langle t \rangle \overset{\mu_1 \times \mathrm{id}}{\twoheadleftarrow} H_1 \times \langle t \rangle \overset{\mu_2 \times \mathrm{id}}{\twoheadleftarrow} H_2 \times \langle t \rangle \overset{\mu_3 \times \mathrm{id}}{\twoheadleftarrow} \cdots \tag{3.3}$$

where each $\langle t \rangle$ is the infinite cyclic group corresponding to the $\mathbb{S}^1$-factor.

The equivalences of (3.1) ensure a ladder diagram between subsequences of (3.2) and (3.3). After relabeling to avoid messy subsequence notation, that diagram has the form:

$$\begin{tikzcd}
H_0 \times \langle t \rangle & & H_1 \times \langle t \rangle \arrow[ll, "\mu_1 \times \mathrm{id}"'] \arrow[dl, "d_1"] & & H_2 \times \langle t \rangle \arrow[ll, "\mu_2 \times \mathrm{id}"'] \arrow[dl, "d_2"] & & H_3 \times \langle t \rangle \cdots \arrow[ll, "\mu_3 \times \mathrm{id}"'] \arrow[dl, "d_3"] \\
& F_0 \times \langle a \rangle \arrow[ul, "u_0"'] & & F_1 \times \langle a \rangle \arrow[ll, "\lambda_1 \times \mathrm{id}"'] \arrow[ul, "u_1"'] & & F_2 \times \langle a \rangle \arrow[ll, "\lambda_2 \times \mathrm{id}"'] \arrow[ul, "u_2"'] & & \cdots \arrow[ll, "\lambda_3 \times \mathrm{id}"']
\end{tikzcd} \tag{3.4}$$

A close look at the homeomorphism between $(\langle j \rangle \backslash X) \times \mathbb{R}$ and $\mathrm{Tor}_j(X)$, as described in [GG, §8], shows that, with appropriate choice of base rays, we may arrange that each u_i takes a to t. Then, by commutativity, each d_i takes t to a, each u_i takes F_i into H_i, and each d_i takes H_i into F_{i-1}. So diagram (3.4) restricts to a diagram of the form

$$\begin{tikzcd}
H_0 & & H_1 \arrow[ll, "\mu_1"'] \arrow[dl] & & H_2 \arrow[ll, "\mu_2"'] \arrow[dl] & & H_3 \ \cdots \arrow[ll, "\mu_3"'] \arrow[dl] \\
& F_0 \arrow[ul] & & F_1 \arrow[ll, "\lambda_1"'] \arrow[ul] & & F_2 \arrow[ll, "\lambda_2"'] \arrow[ul] & & \cdots \arrow[ll, "\lambda_3"']
\end{tikzcd} \tag{3.5}$$

which verifies that pro-$\pi_1(\varepsilon(X))$ is pro-finitely generated free. $\square$

Theorem 3.9. *If a simply connected and finitely dominated Hilbert cube manifold X admits a proper $\mathbb{Z}$-action generated by a homeomorphism $j : X \to X$ that is properly homotopic to id_X, then X is $\mathcal{Z}$-compactifiable.*

Proof. If the action is not cocompact, the previous two lemmas together with Theorem 2.3, ensure that X satisfies the conditions of Theorem 2.9. In the cocompact case, the same lemmas imply that X is inward tame and 2-ended, and that each of those ends is simply connected. In order to use the 1-ended version of Theorem 2.9 provided here, split X into a pair of 1-ended Hilbert cube manifolds and apply the theorem to each end individually. $\square$

Theorem 3.10. *If a simply connected, locally finite polyhedron P is finitely dominated and finite-dimensional, and admits a proper $\mathbb{Z}$-action generated by a homeomorphism $j : P \to P$ that is properly homotopic to id_P, then $P \times \mathbb{I}^{2\cdot\dim P+5}$ is $\mathcal{Z}$-compactifiable.*

Proof. By Theorem 2.7, $j \times \mathrm{id}_{\mathcal{Q}} : P \times \mathcal{Q} \to P \times \mathcal{Q}$ satisfies the hypotheses of Theorem 3.9. An application of Theorem 2.10 completes the proof. $\square$

3.3. **Mapping tori of self-homotopy equivalences.** The goal of this section is an understanding of covering spaces mapping tori of self-homotopy equivalences. Results proved here provide the details missing from the proof of Corollary 1.7 sketched in §1.

Lemma 3.11. *If K is a compact connected ANR with universal cover $\widetilde{K}$, and $f : K \to K$ is homotopy equivalence, then the canonical infinite cyclic cover,* $\mathrm{Tel}_f(K)$, *of* $\mathrm{Tor}_f(K)$ *is proper homotopy equivalent to $K \times \mathbb{R}$ and the universal cover of* $\mathrm{Tor}_f(K)$ *is proper homotopy equivalent to $\widetilde{K} \times \mathbb{R}$.*

Proof. Let $g : K \to K$ be a homotopy inverse for f and $B : K \times [0,1] \to K$ with $B_0 = \mathrm{id}_K$ and $B_1 = fg$. In accordance with Lemma 2.4, our goal in obtaining the initial assertion is to define a map $u : K \times \mathbb{R} \to \mathrm{Tel}_f(K)$, such that there is a proper strong deformation retraction of $\mathcal{M}_{[0,1]}(u)$ onto the domain copy of $K \times \mathbb{R}$. For each integer n, define a function $u_n : K \times [n, n+1] \to \mathcal{M}_{[n,n+1]}(f)$ by the rule:

$$u_n(x,r) = q_{[n,n+1]}(B_{r-n}(g^n(x)), r), \text{ when } n \geq 0$$

and

$$u_n(x,r) = q_{[n,n+1]}\left(f^{-n}(x), r\right), \text{ when } n < 0.$$

Here it is understood that $g^0 = \mathrm{id}_K$.

Note that

$$u_{-1}(x,0) = q_{[-1,0]}(f(x), 0)), \text{ while } u_0(x,0) = q_{[0,1]}(B_0(x), 0) = q_{[0,1]}(x,0)$$

and for each integer $n \geq 1$,

$$u_{n-1}(x,n) = q_{[n-1,n]}\left(B_1\left(g^{n-1}(x)\right)\right) = q_{[n-1,n]}\left(fgg^{n-1}(x), n\right) = q_{[n-1,n]}(fg^n(x), n)$$

and

$$u_n(x,n) = q_{[n,n+1]}(B_0(g^n(x)), n) = q_{[n,n+1]}(g^n(x), n).$$

Similarly, for each for each integer $n \leq -1$,

$$u_{n-1}(x,n) = q_{[n-1,n]}\left(f^{-(n-1)}(x), n)\right) = q_{[n-1,n]}\left(f(f^{-n}(x), n)\right)$$

and

$$u_n(x,n) = q_{[n,n+1]}\left(f^{-n}(x), n)\right).$$

It follows that the u_n can be glued together to obtain a proper map $u : K \times \mathbb{R} \to \mathrm{Tel}_f(K)$. See Figure 1.

Claim. *There is a proper strong deformation retraction of $\mathcal{M}_{[0,1]}(u)$ onto $K \times \mathbb{R}$.*

First note that, since u respects $\mathbb{R}$-coordinates, the natural projections $K \times \mathbb{R} \to \mathbb{R}$ and $p : \mathrm{Tel}_f(K) \to \mathbb{R}$ can be extended to a projection $\widehat{p} : \mathcal{M}_{[0,1]}(u) \to \mathbb{R}$ with the property that each point preimage $\widehat{p}^{-1}(r)$ is a mapping cylinder C_r of a map from $K \times \{r\}$ to K_r. Indeed, for an integer $n \geq 0$, C_n is the mapping cylinder of fg^n and for an integer $n < 0$, C_n is the mapping cylinder of $f^{-(n-1)}$. So each C_n is a mapping cylinder of a homotopy equivalence—a fact that will be useful later. (In fact, each C_r is a mapping cylinder of a homotopy equivalence, but this fact will only be used for integral values of r.) Note also that $\mathcal{M}_{[0,1]}(u)$ may be viewed as a countable union $\bigcup_{n\in\mathbb{Z}} \mathcal{M}_{[0,1]}(u_n)$, where each $\mathcal{M}_{[0,1]}(u_n)$ intersects $\mathcal{M}_{[0,1]}(u_{n-1})$ in C_n.

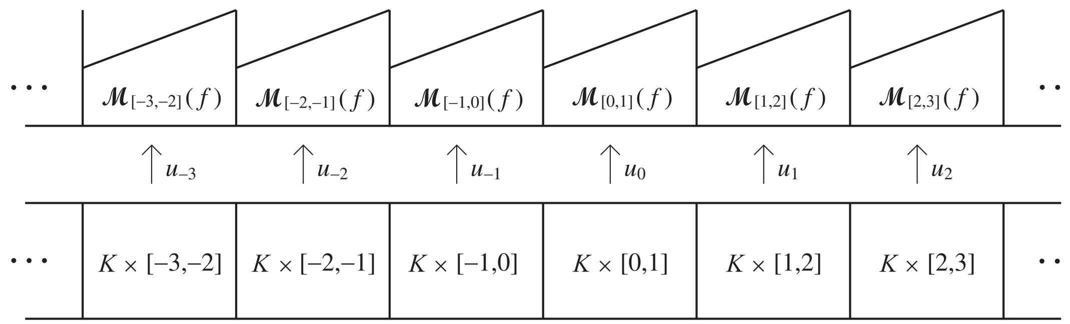

FIGURE 1. The map $u : K \times \mathbb{R} \to \mathrm{Tel}_f(K)$.

Subclaim. *For each n, $\mathcal{M}_{[0,1]}(u_n)$ strong deformation retracts onto the subset $C_n \cup (K \times [n, n+1])_1 \cup C_{n+1}$.*

It suffices to show that $C_n \cup (K \times [n, n+1])_1 \cup C_{n+1} \hookrightarrow \mathcal{M}(u_n)$ is a homotopy equivalence. Since C_n and C_{n+1} are mapping cylinders of homotopy equivalences, each strong deformation retracts onto its domain end, so $(K \times [n, n+1])_1 \hookrightarrow C_n \cup (K \times [n, n+1])_1 \cup C_{n+1}$ is a homotopy equivalence; therefore, it is enough to show that $(K \times [n, n+1])_1 \hookrightarrow \mathcal{M}_{[0,1]}(u_n)$ is a homotopy equivalence. Note that the inclusions $K_n \hookrightarrow C_n$, $K_n \hookrightarrow \mathcal{M}_{[n,n+1]}(f)$ and $\mathcal{M}_{[n,n+1]}(f) \hookrightarrow \mathcal{M}_{[0,1]}(u_n)$ are all homotopy equivalences, since each subspace is the range end of a corresponding mapping cylinder. It follows that $C_n \hookrightarrow \mathcal{M}_{[0,1]}(u_n)$ is a homotopy equivalence, and since $K \times \{n\} \hookrightarrow C_n$ is a homotopy equivalence it follows that $K \times \{n\} \hookrightarrow \mathcal{M}_{[0,1]}(u_n)$, and hence, $(K \times [n, n+1])_1 \hookrightarrow \mathcal{M}_{[0,1]}(u_n)$ is a homotopy equivalence. The subclaim follows.

To complete the claim, first properly strong deformation retract $\mathcal{M}_{[0,1]}(u)$ onto $(K \times \mathbb{R})_1 \cup \left(\bigcup_{n\in\mathbb{Z}} C_n\right)$ using the union of the strong deformation retractions provided by the subclaim. Follow that by a strong deformation of $(K \times \mathbb{R})_1 \cup \left(\bigcup_{n\in\mathbb{Z}} C_n\right)$ onto $(K \times \mathbb{R})_1$ obtained by individually strong deformation retracting each C_n onto its domain end.

Since proper homotopy equivalences lift to proper homotopy equivalences of their universal covers, the assertion about universal covers immediately from the initial assertion. $\square$

Theorem 3.12. *If K is a compact connected ANR with a finitely dominated universal cover, $f : K \to K$ is a homotopy equivalence, and $\widetilde{\mathrm{Tor}_f(K)}$ is the universal cover of the mapping torus of f, then $\widetilde{\mathrm{Tor}_f(K)} \times \mathcal{Q}$ is $\mathcal{Z}$-compactifiable.*

Proof. By combining Lemma 3.11 with Lemma 3.1, Lemma 3.4, and Theorem 2.3, we conclude that $\widetilde{\mathrm{Tor}_f(K)} \times \mathcal{Q}$ satisfies all hypotheses of Chapman and Siebenmann's $\mathcal{Z}$-compactification theorem. An alternative argument can be obtained by using Lemma 3.8 in place of Lemma 3.4. $\square$

Theorem 3.13. *If K is a finite polyhedron with finitely dominated universal cover, $f : K \to K$ is a simplicial homotopy equivalence, and $\widetilde{\mathrm{Tor}_f(K)}$ is the universal cover of the mapping torus of f, then $\widetilde{\mathrm{Tor}_f(K)} \times \mathbb{I}^{2\cdot\dim K+7}$ is $\mathcal{Z}$-compactifiable.*

Proof. Apply Theorem 3.12 and Ferry's Theorem 2.10 to the $(\dim K + 1)$-dimensional polyhedron $\widetilde{\mathrm{Tor}_f(K)}$. □

3.4. **Spaces that are simply connected at infinity.** The key result about Hilbert cube manifolds that are simply connected at infinity is our easiest application of Theorem 2.9; it can be found in Chapman and Siebenmann's original work. For completeness, we include a sketch of their proof.

Theorem 3.14 ([CS, Cor. to Th.8])**.** *If X is a Hilbert cube manifold that is simply connected at infinity and $H_*(X;\mathbb{Z})$ is finitely generated, then X is $\mathcal{Z}$-compactifiable.*

Sketch of Proof. Due to the triviality of pro-$\pi_1(\varepsilon(X))$, we need only show that X is inward tame. If N is a clean neighborhood of infinity, then $\mathrm{Bd}_X N$ is homotopy equivalent to a finite complex. Since $H_i(\mathrm{Bd}_X N;\mathbb{Z})$ and $H_i(X;\mathbb{Z})$ are both finitely generated for all i, and eventually trivial, the Mayer-Vietoris sequence

$$\cdots \to H_i(\mathrm{Bd}_X N;\mathbb{Z}) \to H_i\left(\overline{X-N};\mathbb{Z}\right) \oplus H_i(N;\mathbb{Z}) \to H_i(X;\mathbb{Z}) \to \cdots$$

shows that the same is true for $H_i(N;\mathbb{Z})$. Furthermore, the simple connectivity at infinity of X, together with standard techniques from Hilbert cube manifold topology, ensure the existence of arbitrarily small simply connected N. Since a simply connected complex with finitely generated $\mathbb{Z}$-homology necessarily has finite homotopy type (see [Sp, p.420]), it follows that X is inward tame. □

Theorem 3.15. *If P is a finite-dimensional, locally finite polyhedron that is simply connected at infinity and $H_*(P;\mathbb{Z})$ is finitely generated, then $P \times \mathbb{I}^{2\cdot\dim P+5}$ is $\mathcal{Z}$-compactifiable.*

Proof. Apply Theorems 2.7, 3.14, 2.10. □

4. Proofs of the main theorems

In this section we assemble the above ingredients into proofs of our main results: Theorems 1.5, 1.6, and 1.8 and Corollary 1.7.

Proof of Theorem 1.5. Since $\widetilde{K}$ is contractible, both X and Y are also contractible. By Lemmas 3.2 and 3.4, $X \times Y$ is inward tame and 1-ended with pro-$\pi_1(X \times Y, r)$ that is pro-finitely generated free, and since $\widetilde{K} \overset{p}{\simeq} X \times Y$, each of these properties is inherited by $\widetilde{K}$. Applying Theorems 2.7, 2.3, and 2.9 in the usual way provides a $\mathcal{Z}$-compactification of $\widetilde{K} \times \mathcal{Q}$, and since $\dim \widetilde{K} = \dim K < \infty$, Theorem 2.10 provides a $\mathcal{Z}$-compactification of the ER $\widetilde{K} \times \mathbb{I}^{2\cdot\dim K+5}$. □

Proof of Theorem 1.6. By Lemmas 3.7 and 3.8, X is inward tame, and either: 2-ended and simply connected at each end; or 1-ended with pro-finitely generated free fundamental group at infinity. By proper homotopy invariance, the same is true for $\widetilde{K}$, so by the usual argument, $\widetilde{K} \times \mathcal{Q}$ is $\mathcal{Z}$-compactifiable. Another application of Theorem 2.10 provides a $\mathcal{Z}$-compactification of $\widetilde{K} \times \mathbb{I}^{2\cdot\dim K+5}$. □

Remark 5. In the special case, where X (or $\widetilde{K}$) admits a cocompact $\mathbb{Z}$-action, the above argument is overkill. There, since X is contractible, $\langle j \rangle \backslash X$ is homotopy equivalent to a circle; and since $\langle j \rangle \backslash X$ is compact, a homotopy equivalence $f : \langle j \rangle \backslash X \to S^1$ lifts to a proper homotopy equivalence $X \overset{p}{\simeq} \mathbb{R}$. It is then straightforward to show that the 2-point compactifications of X and $\widetilde{K}$ are themselves $\mathcal{Z}$-compactifications.

Proof of Theorem 1.8. Since G is type F, each nontrivial element has infinite order, so we may apply [GG, Th.1.4] to conclude that G is either simply connected at infinity or G is virtually a surface group. In other words, if K is a finite $\mathrm{K}(G,1)$ complex, then $\widetilde{K}$ is either simply connected at infinity, or $\widetilde{K}$ is the universal cover of the corresponding finite $\mathrm{K}(H,1)$ complex $H\backslash\widetilde{K}$, where H is a finite index subgroup of G and $H \cong \pi_1(S)$, where S is a closed surface with infinite fundamental group.

In the case where $\widetilde{K}$ is simply connected at infinity, we may apply Theorem 3.14 to conclude that $\widetilde{K} \times \mathcal{Q}$ is $\mathcal{Z}$-compactifiable, and hence $\widetilde{K} \times \mathbb{I}^{2\cdot\dim K+5}$, admits the desired $\mathcal{Z}$-compactification.

In the second case, we may conclude that $\widetilde{K} \overset{p}{\simeq} \widetilde{S} \approx \mathbb{R}^2$. It follows that $\widetilde{K}$ is 1-ended and inward tame, with pro-$\pi_1\left(\varepsilon\left(\widetilde{K}\right)\right)$ stably isomorphic to $\mathbb{Z}$. By Theorem 2.3, $\widetilde{K} \times \mathcal{Q}$ satisfies the hypotheses of Theorem 2.9, and is therefore $\mathcal{Z}$-compactifiable. Another application of Theorem 2.10 completes the proof. □

As for Corollary 1.7, all details missing from the sketch given in §1 were provided in §3.3.

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